Exact solution of mean geodesic distance for Vicsek fractals

Zhongzhi Zhang[1,2],* Shuigeng Zhou[1,2],† Lichao Chen [1,2], Ming Yin[1,2], and Jihong Guan[3]
[1] *Department of Computer Science and Engineering, Fudan University, Shanghai 200433, China*
[2] *Shanghai Key Lab of Intelligent Information Processing, Fudan University, Shanghai 200433, China and*
[3] *Department of Computer Science and Technology, Tongji University, 4800 Cao'an Road, Shanghai 201804, China*
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The Vicsek fractals are one of the most interesting classes of fractals and the study of their structural properties is important. In this paper, the exact formula for the mean geodesic distance of Vicsek fractals is found. The quantity is computed precisely through the recurrence relations derived from the self-similar structure of the fractals considered. The obtained exact solution exhibits that the mean geodesic distance approximately increases as an exponential function of the number of nodes, with the exponent equal to the reciprocal of the fractal dimension. The closed-form solution is confirmed by extensive numerical calculations.

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The concept of fractals plays an important role in characterizing the features of complex systems in nature, since many objects in the real world can be modeled by fractals [1]. In the last two decades, a great deal of activity has been concentrated on the studies of fractals [2, 3]. It has been shown [4, 5, 6, 7, 8, 9, 10, 11] that regular fractals capture important aspects of critical percolation clusters, aerogels, amorphous solids, and unusual phase transition in the Ising model. Among various regular fractals, the Vicsek fractals [12] are a class of typical candidates for exact mathematical ones and have received much attention. A variety of structural and dynamical properties of Vicsek fractals have been investigated in much detail, including eigenvalue spectrum [13], eigenstates [14], Laplacian spectrum [15], random walks [16], diffusion [17], and so on. The results of these investigations uncovered many unusual and exotic features of Vicsek fractals.

A central issue in the study of complex systems is to understand how their dynamical behaviors are influenced by underlying geometrical and topological properties [18, 19]. Among many fundamental structural characteristics [20], mean geodesic distance is an important topological feature of complex systems that are often described by graphs (or networks) where nodes (vertices) represent the component units of systems and links (edges) stand for the interactions between them [21, 22]. Mean geodesic distance is defined as the mean length of the shortest paths between all pairs of nodes. It has been well established that mean geodesic distance directly relates to many aspects of real systems, such as signal integrity in communication networks, the propagation of beliefs in social networks or of technology in industrial networks. Recent studies indicated that a number of other dynamical processes are also relevant to mean geodesic distance, including disease spreading [23], random walks [24], navigation [25], to name but a few. Thus far great efforts have been made to valuate and understand the mean geodesic distance of different systems [26, 27, 28, 29, 30, 31].

Despite the importance of this structural property, to the best of our knowledge, the rigorous computation for the mean geodesic distance of Vicsek fractals has not been addressed. To fill this gap, in this present paper we investigate this interesting quantity analytically. We derive an exact formula for the mean geodesic distance characterizing the Vicsek fractals. The analytic method is on the basis of an algebraic iterative procedure obtained from the self-similar structure of Vicsek fractals. The obtained precise result shows that the mean geodesic distance exponentially with the number of nodes. Our research opens the way to theoretically study the mean geodesic distance of regular fractals and deterministic networks [32, 33, 34]. In particularly, our exact solution gives insight different from that afforded by the approximate solution of stochastic fractals.

The classical Vicsek fractals are constructed iteratively [12, 15]. We denote by $V_{f,t}$ ($t \geq 0$, $f \geq 2$) the Vicsek fractals after t generations. The construction starts from ($t = 0$) a star-like cluster consist of $f + 1$ nodes arranged in a cross-wise pattern, where f peripheral nodes are connected to a central node. This corresponds to $V_{f,0}$. For $t \geq 1$, $V_{f,t}$ is obtained from $V_{f,t-1}$. To obtain $V_{f,1}$, we generate f replicas of $V_{f,0}$ and arrange them around the periphery of the original $V_{f,0}$, then we connect the central structure by f additional links to the corner copy structures. These replication and connection steps are repeated t times, with the needed Vicsek fractals obtained in the limit $t \to \infty$, whose fractal dimension is $\frac{\ln(f+1)}{\ln 3}$. In Fig. 1, we show schematically the structure of $V_{3,2}$. According to the construction algorithm, at each time step the number of nodes in the systems increase by a factor of $f + 1$, thus, we can easily know that the total number of nodes (network order) of $V_{f,t}$ is $N_t = (f + 1)^{t+1}$.

After introducing the Vicsek fractals, we now investi-

*Electronic address: zhangzz@fudan.edu.cn
†Electronic address: sgzhou@fudan.edu.cn

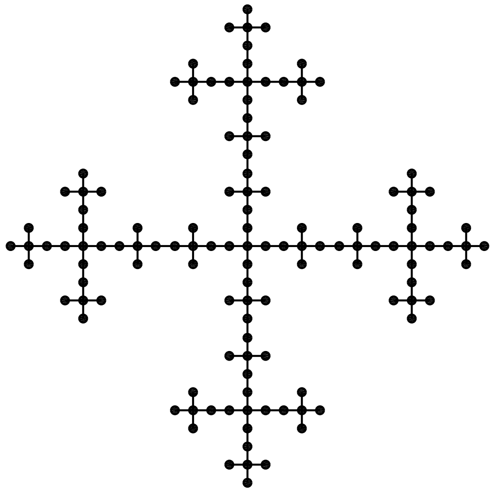

FIG. 1: Illustration of a particular Vicsek fractal $V_{3,2}$.

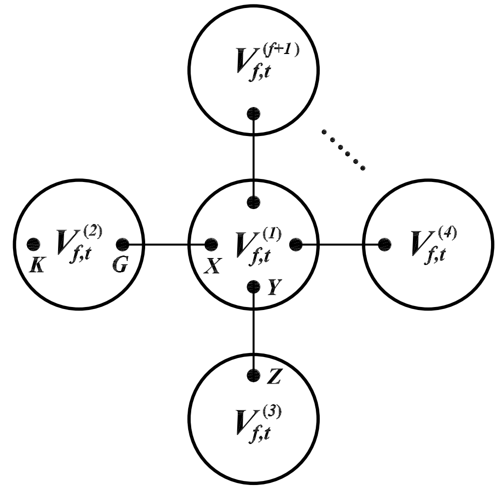

FIG. 2: A schematic illustration of the iterative construction for $V_{f,t+1}$, which is obtained by joining $f+1$ copies of $V_{f,t}$ denoted as $V_{f,t}^{(1)}$, $V_{f,t}^{(2)}$, $\cdots$, $V_{f,t}^{(f)}$, and $V_{f,t}^{(f+1)}$, respectively.

gate analytically the mean geodesic distance between all the node pairs in the fractals. We represent all the shortest path lengths of $V_{f,t}$ as a matrix in which the entry d_{ij} is the geodesic distance from node i to node j, where geodesic distance is the path connecting two nodes with minimum length. The maximum value D_t of d_{ij} is called the diameter of $V_{f,t}$. A measure of the typical separation between two nodes in $V_{f,t}$ is given by the mean geodesic distance L_t defined as the mean of geodesic lengths over all couples of nodes:

$$L_t = \frac{S_t}{N_t(N_t-1)/2}\,, \tag{1}$$

where

$$S_t = \sum_{i\in V_{f,t},\, j\in V_{f,t},\, i\neq j} d_{ij} \tag{2}$$

denotes the sum of the geodesic distances between two nodes over all pairs.

We continue by exhibiting the procedure of the determination of the total distance and present the recurrence formula, which allows us to obtain S_{t+1} of the $t+1$ generation from S_t of the t generation. By construction, the fractal $V_{f,t+1}$ is obtained by the juxtaposition of $f+1$ copies of $V_{f,t}$ that are consecutively labeled as $V_{f,t}^{(1)}$, $V_{f,t}^{(2)}$, $\cdots$, $V_{f,t}^{(f+1)}$, see Fig. 2. This obvious self-similar structure allows us to calculate S_t analytically. It is easy to see that the total distance S_{t+1} satisfies the recursion relation

$$S_{t+1} = (f+1)\,S_t + \Theta_t, \tag{3}$$

where Θ_t is the sum over all shortest path length whose endpoints are not in the same $V_{f,t}$ branch. The solution of Eq. (3) is

$$S_t = (f+1)^t\,S_0 + \sum_{m=0}^{t-1}\left[(f+1)^{t-m-1}\Theta_m\right]. \tag{4}$$

Thus, all that is left to obtain S_t is to compute Θ_m.

The paths that contribute to Θ_t must all go through at least one of the $2f$ edge nodes (such as G, X, Y, and Z in Fig. 2) at which the different $V_{f,t}$ branches are connected. The analytical expression for Θ_t, named the crossing path length, can be derived as below.

Denote $\Theta_t^{\alpha,\beta}$ as the sum of all shortest paths with endpoints in $V_{f,t}^{(\alpha)}$ and $V_{f,t}^{(\beta)}$. For convenience, we denote by $V_{f,t}^{(1)}$ the central branch of $V_{f,t+1}$. According to whether or not the two branches are adjacent, we sort the crossing path length $\Theta_t^{\alpha,\beta}$ into two classes: $\Theta_t^{1,\phi}$ ($\phi>1$), $\Theta_t^{\varphi,\theta}$ ($\varphi>1$, $\theta>1$, and $\varphi\neq\theta$). For any two crossing paths in the same class, they have identical length. Therefore, in the following computation of Θ_t, we will only consider $\Theta_t^{1,2}$ and $\Theta_t^{2,3}$. The total sum Θ_t is then given by

$$\Theta_t = f\times\Theta_t^{1,2} + \binom{f}{2}\times\Theta_t^{2,3}. \tag{5}$$

To calculate the crossing path length $\Theta_t^{1,2}$ and $\Theta_t^{2,3}$, we give the following definition and notations. We define external nodes of $V_{f,t}$ as the nodes that will be linked to one of its copes at step $t+1$ to form $V_{f+1,t}$. Let d_t denote the sum of length of the path from an external node of $V_{f,t}$ to all nodes in $V_{f,t}$ including the external node itself. We assume that the two branches $V_{f,t}^{(1)}$ and $V_{f,t}^{(2)}$ are connected at two nodes X and G, which separately belong to $V_{f,t}^{(1)}$ and $V_{f,t}^{(2)}$, and that $V_{f,t}^{(1)}$ and $V_{f,t}^{(3)}$ are linked to each other at two nodes Y and Z that are in $V_{f,t}^{(1)}$ and $V_{f,t}^{(3)}$, respectively.

In order to determine d_t, we should compute the diameter D_t of $V_{f,t}$ first. By construction, one can see that the diameter D_t equals the path length between arbitrary pair of external nodes of $V_{f,t}$. Thus, we have the

following recursive relation:

$$D_{t+1} = 3\,D_t + 2. \tag{6}$$

Considering the initial condition $D_0 = 2$, Eq. (6) is solved inductively to obtain

$$D_t = 3^{t+1} - 1, \tag{7}$$

which is independent of f.

We now calculate the quantity d_{t+1}. Let K denote the external node of $V_{f,t+1}$, which is in the branch $V_{f,t}^{(2)}$. By definition, d_{t+1} can be given by the sum

$$\begin{aligned} d_{t+1} &= \sum_{j\in V_{f,t+1}} d_{Kj} \\ &= \sum_{u\in V_{f,t}^{(2)}} d_{Ku} + \sum_{v\in V_{f,t}^{(1)}} d_{Ku} + (f-1)\sum_{w\in V_{f,t}^{(3)}} d_{Kw} \\ &= d_t + \sum_{v\in V_{f,t}^{(1)}} d_{Ku} + (f-1)\sum_{w\in V_{f,t}^{(3)}} d_{Kw}. \end{aligned} \tag{8}$$

We denote the second and third terms in Eq. (8) by g_t and q_t, respectively. Thus, $d_{t+1} = d_t + g_t + q_t$. The quantity g_t is evaluated as follows:

$$\begin{aligned} g_t &= \sum_{v\in V_{f,t}^{(1)}} (d_{KG} + d_{GX} + d_{Xu}) \\ &= d_t + N_t \times (D_t + 1), \end{aligned} \tag{9}$$

where $d_{KX} = D_t$ and $d_{GX} = 1$ were used. Analogously,

$$\begin{aligned} q_t &= (f-1)\sum_{w\in V_{f,t}^{(3)}} (d_{KG} + d_{GX} + d_{XY} + d_{YZ} + d_{Zw}) \\ &= (f-1)\left[d_t + N_t \times 2(D_t+1)\right]. \end{aligned} \tag{10}$$

With Eqs. (9) and (10), Eq. (8) becomes

$$d_{t+1} = (f+1)d_t + (2f-1)\times N_t \times (D_t+1). \tag{11}$$

Using $N_t = (f+1)^{t+1}$, $D_t = 3^{t+1} - 1$ and $d_0 = 2f-1$, Eq. (11) is resolved by induction

$$d_t = \frac{1}{2}(2f-1)\left(3^{t+1}-1\right)(1+f)^t. \tag{12}$$

With above obtained results, we can determine the length of crossing paths $\Theta_t^{1,2}$ and $\Theta_t^{2,3}$, which can be expressed in terms of the previously explicitly determined quantities. By definition, $\Theta_t^{1,2}$ is given by the sum

$$\begin{aligned} \Theta_t^{1,2} &= \sum_{i\in V_{f,t}^{(1)},\, j\in V_{f,t}^{(2)}} d_{ij} \\ &= \sum_{i\in V_{f,t}^{(1)},\, j\in V_{f,t}^{(2)}} (d_{iX} + d_{XG} + d_{Gj}) \\ &= N_t \sum_{i\in V_{f,t}^{(1)}} d_{iX} + (N_t)^2 + N_t \sum_{j\in V_{f,t}^{(2)}} d_{Gj} \\ &= 2N_t \sum_{i\in V_{f,t}^{(1)}} d_{iX} + (N_t)^2, \end{aligned} \tag{13}$$

where we have used the equivalence relation $\sum_{i\in V_{f,t}^{(1)}} d_{iX} = \sum_{j\in V_{f,t}^{(2)}} d_{Gj}$.

Proceeding similarly,

$$\begin{aligned} \Theta_t^{2,3} &= \sum_{i\in V_{f,t}^{(2)},\, j\in V_{f,t}^{(3)}} d_{ij} \\ &= \sum_{i\in V_{f,t}^{(2)},\, j\in V_{f,t}^{(3)}} (d_{iG} + d_{GX} + d_{XY} + d_{YZ} + d_{Zj}) \\ &= 2N_t \sum_{i\in V_{f,t}^{(2)}} d_{iG} + 2\,(N_t)^2 + (N_t)^2\, d_{XY} \\ &= 2N_t \sum_{i\in V_{f,t}^{(2)}} d_{iG} + (N_t)^2 (D_t + 2). \end{aligned} \tag{14}$$

Inserting Eqs. (13) and (14) into Eq. (5), we have

$$\Theta_t = (f^2+f)N_t\, d_t + f^2\,(N_t)^2 + \frac{f(f-1)}{2}(N_t)^2\, D_t. \tag{15}$$

Substituting Eq. (15) into Eq. (4) and using the initial value $S_0 = f^2$, we can obtain the exact expression for the total distance

$$\begin{aligned} S_t &= \frac{1}{6f+4}(f+1)^t\Big[f^2(f+1)^t\left(3^{t+1}+1\right) \\ &+ 3\left(3^{t+1}-1\right)f^3(f+1)^t + 4\left((f+1)^t-1\right) \\ &- 2f\left(3^{1+t}(f+1)^t - 4(1+f)^t + 1\right)\Big] \end{aligned} \tag{16}$$

Then the analytic expression for mean geodesic distance can be obtained as

$$\begin{aligned} L_t &= \frac{1}{(3f^2+5f+2)[(f+1)^t-1]}\Big[f^2(f+1)^t\left(3^{t+1}+1\right) \\ &+ 3\left(3^{t+1}-1\right)f^3(f+1)^t + 4\left((f+1)^t-1\right) \\ &- 2f\left(3^{1+t}(f+1)^t - 4(1+f)^t + 1\right)\Big]. \end{aligned} \tag{17}$$

In the infinite system size, i.e., $t \to \infty$

$$L_t \sim 3^{t+1} = (N_t)^{\frac{\ln 3}{\ln(f+1)}}, \tag{18}$$

where the exponent $\frac{\ln 3}{\ln(f+1)}$ is equal to the reciprocal of the fractal dimension. Thus, the mean geodesic distance grows exponentially with increasing size of the system. In contrast to many recently studied network models mimicking real-life systems in nature and society [21, 22], the Vicsek fractals are not small worlds despite of the fact that these fractals show similarity (fractality) observed in many real-world systems.

We have checked our analytic result against numerical calculations for different f and various t. In all the cases we obtain a complete agreement between our theoretical formula and the results of numerical investigation, see Fig. 3.

To sum up, in complex systems the mean geodesic distance plays an important role. It has a profound impact

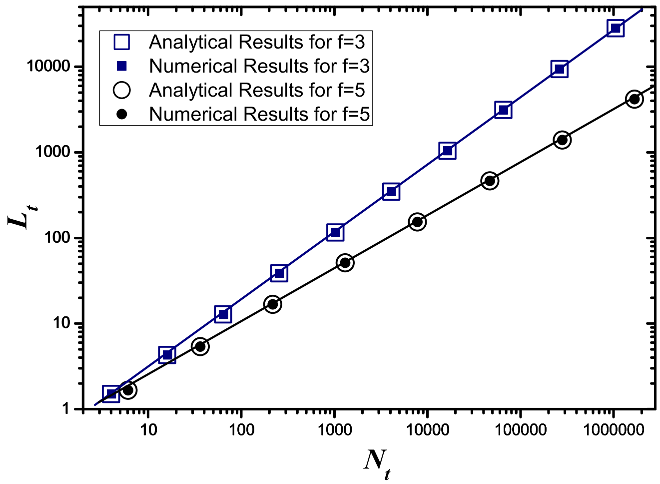

FIG. 3: Mean geodesic distance L_t versus network order N_t on a log-log scale. The solid lines are guides to the eyes.

on a variety of crucial fields, such as information processing, disease or rumor transmission, network designing and optimization. In this paper, we have derived analytically the solution for the mean geodesic distance of Vicsek fractals which have been attracting much research interest. We found that in the infinite network size limit the mean geodesic distance scales exponentially with the number of nodes. Our analytical technique could guide and shed light on related studies for deterministic fractals and network models by providing a paradigm for calculating the mean geodesic distance. Moreover, as a guide to and a test of approximate methods, we believe our vigorous solution can prompt the studies on random fractals.

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